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Abstract: This paper deals with the detection of emerging plane cracks, by using boundary measurements. An identifiability result (uniqueness of the solution) is first proved. Then, we look at the stability of this solution with respect to the measurement. A weak stability result is proved, as well as a local Lipshitz stability result for straight cracks, by using domain-derivative techniques.

Key-words: geometrical inverse problems, crack detection, identifiability, stability, Lipschitz stability.

(Résumé : tsvp)

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Sur le problème inverse relatif aux fissures planes débouchantes

Résumé : On s'intéresse dans ce travail à la détection de fissures planes débouchantes au moyen de mesures de frontière. On prouve d'abord un résultat d'identifiabilité (unicité de la solution). On s'intéresse ensuite à la stabilité de la solution par rapport à la mesure. Un résultat de stabilité faible est d'abord montré, puis un résultat de stabilité locale lipschitzienne est obtenu pour les fissures droites, en faisant usage de techniques de dérivation par rapport aux domaines.

Mots-clé : problèmes inverse géométriques, détection de fissures, identifiabilité, stabilité, stabilité lipschitzienne.

1 Introduction

An industrial and theoretical important problem is the determination of cracks by overdetermined data. It consists in finding the shape and the location of fractures inside a body occupying a domain Ω by measuring the temperature on the boundary, a steady state heat flux being prescribed on a part of this boundary.
In order to solve this inverse geometrical problem, one has first to prove that it is well posed in Hadamard's sense, which means that it has a unique solution, and that a small perturbation of the data involve small perturbations on the unknown, which are the shape and the location of the crack. Furthermore, it is of great interest to define some inversion process to find out the defect by use of the measure got on the boundary.
Up to our knowledge, only a few theoretical works exist in this area : the uniqueness (*identifiability*) result for a buried single crack has been proved by Friedman and Vogelius [12, 1989], provided that two special boundary measurements are available. In the same paper, a partial stability result has been established. For the same kind of problem, Alessandrini, Beretta and Vessella [3] proved a Lipshitz stability result for linear cracks in 1993. On another hand, a constructive explicit process has been developed by Andrieux and Ben Abda [4] in 1992. This process appears to be also, as a matter of fact, an identifiability result, valid for linear (2D) or planar (3D) cracks. In the case of a collection of cracks, Bryan and Vogelius [7, 1992], followed by Alessandrini, and Diaz Valenzuela [2, 1996], proved uniqueness results.
For emerging cracks, Elcrat, Isakov and Neculoiu [11, 1995] gave a uniqueness result. We are here dealing with the 2D inverse crack problem, emerging in a known point of the external boundary. The first section is devoted to the proof of a uniqueness result, provided that the measure corresponds to a special heat flux we shall explicitely construct. As for the second section, it is devoted to stability results : the weak stability is proved by use of the compactness of the set of admissible cracks. Then a local Lipshitz stability result is established by using the Lagrangian derivatives with respect to the domain.

2 Identifiability

Let Ω denote the domain occupied by the body, which is supposed to contain *exactly* one crack σ emerging in a known point S of the boundary $\partial\Omega$. In the whole paper, a crack is understood as a $\mathcal{C}^2$ non self-intersecting curve. The boundary $\partial\Omega$ is parametrized by the arc length s, the point S being the origin of this parametrization.
Consider three points P, Q, R of the external boundary $\partial\Omega$, such that

$$0 < s(R) < s(P)$$

and let us define the flux used by :

$$\phi(x) = \begin{cases} 1 & on & RQ \\ -\frac{|RQ|}{|PQ|} & on & QP \\ 0 & elsewhere & \end{cases}$$

The direct problem is therefore given by :

$$\begin{cases} \Delta u_\sigma & = & 0 & in & \Omega_\sigma \\ \frac{\partial u_\sigma}{\partial n} & = & 0 & on & \sigma \\ \frac{\partial u_\sigma}{\partial n} & = & \phi & on & \partial\Omega \end{cases} \tag{1}$$

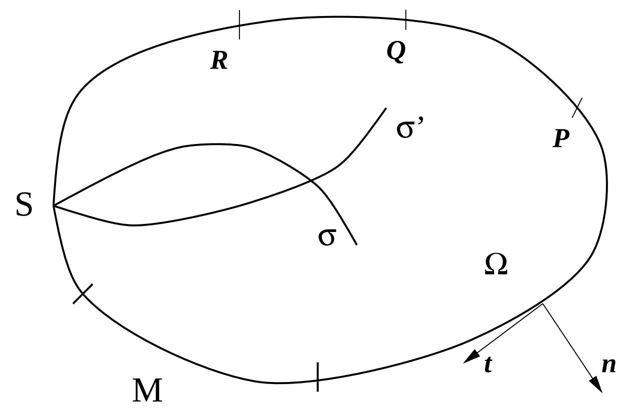

Figure 1: *The cracked domain*

The condition $\int_{\partial\Omega} \phi = 0$ insures the existence of solutions to the above problem, while to insure the uniqueness of such a solution, one has to add for example the condition $\int_{\partial\Omega} u_\sigma = 0$. M is the part of the boundary $\partial\Omega$, with positive measure, on which the temperature has been measured.

Theorem 1 (identifiability) *Let σ and $\sigma\prime$ be two emerging cracks in Ω, ending at the same point S of the boundary $\partial\Omega$. If they both lead to the same measurement of the temperature, for the prescribed flux ϕ defined previously, then $\sigma = \sigma\prime$.*

Proof : Let u_σ be the solution corresponding to the crack σ, and $u_{\sigma\prime}$ the one corresponding to $\sigma\prime$. Then, $u_\sigma \in \Omega_\sigma = \Omega\backslash\sigma$, and we define τ_σ as the vector field ∇u_σ, which is divergence-free in $L^2(\Omega_\sigma)$. Thus, τ_σ has two normal traces, denoted by ${\tau_\sigma}^+$ and${\tau_\sigma}^-$, on the upper and lower side of σ.

Since

$$\nabla u_\sigma . n^+ = \nabla u_\sigma . n^- = 0$$

then

$$\tau_\sigma . n^+ = \tau_\sigma . n^- = 0$$

It turns out then that $div\,\tau_\sigma$, considered as a distribution, is vanishing in the whole domain Ω, and therefeore there exists some scalar function $\omega_\sigma \in H^1(\Omega)$, such that :

$$\tau_\sigma = -(\nabla\omega_\sigma)^\perp = \left(\frac{\partial\omega_\sigma}{\partial x_2}, -\frac{\partial\omega_\sigma}{\partial x_1}\right)$$

where x_1 and x_2 are the cartesian coordinates. The function ω_σ is unique, up to a constant. Furthermore, one has :

$$\frac{\partial\omega_\sigma}{\partial t} = -(\nabla\omega_\sigma)^\perp . n = \phi \quad on \quad \partial\Omega$$

and thus

$$\omega_\sigma = K_\sigma \quad on \quad \sigma.$$

where K_σ is some constant. Thus, $\omega_\sigma \in H^1(\Omega)$, since it is continuous across the crack σ, and it is solution of the following problem :

$$\left\{\begin{array}{rcll} \Delta\omega_\sigma & = & 0 & in \quad \Omega_\sigma \\ \omega_\sigma & = & K_\sigma & on \quad \sigma \\ \omega_\sigma & = & \varphi & on \quad \partial\Omega \end{array}\right. \tag{2}$$

where φ is a fonction defined on the external boundary $\partial\Omega$, which derivative with respect to the arclength s is the prescribed flux ϕ. Obviously then, $\varphi(S) = K_\sigma$. $\omega_{\sigma\prime}$ is, in the same way, solution of the following problem :

$$\begin{cases} \Delta\omega_{\sigma\prime} &= 0 & in & \Omega_{\sigma\prime} \\ \omega_{\sigma\prime} &= K_{\sigma\prime} & on & \sigma\prime \\ \omega_{\sigma\prime} &= \varphi & on & \partial\Omega \end{cases} \tag{3}$$

Since ω_σ and $\omega_{\sigma'}$ are defined up to a constant, one can suppose that $K_\sigma = K_{\sigma'}$. Let us now denote by ω the field :

$$\omega = \omega_\sigma - \omega_{\sigma\prime}$$

which is harmonic in the domain $\Omega\backslash(\sigma \cup \sigma\prime)$, and satisfies :

$$\omega = 0 \quad on \quad M$$

as well as :

$$\frac{\partial\omega}{\partial n} = -(\nabla\omega_\sigma)^\perp . t + (\nabla\omega_{\sigma\prime})^\perp . t = \tau_{\sigma\prime}. t - \tau_\sigma . t = \frac{\partial}{\partial t}(u_{\sigma'} - u_\sigma) = 0 \quad on \quad M$$

since u_σ and $u_{\sigma'}$ have the same trace on M, which is the measured temperature. It comes then, by the Holmgren's uniqueness theorem, that $\omega \equiv 0$ in the exterior connected component of $\Omega\backslash(\sigma \cup \sigma')$, that we shall denote by Ω_{ext}, and therefore on its boundary, which contains $\sigma \cup \sigma'$. By the specific choice we made of φ (actually of ϕ), $\varphi(S)$ is the minimum of φ on the external boundary $\partial\Omega$, and φ is taking this minimum value on the arc PR.
Let us now suppose that $\sigma \neq \sigma'$. Then, there exists some point $z \in \sigma'\backslash\sigma$ (or $\sigma'\backslash\sigma$) which is an interior point to Ω_σ (or to $\Omega_{\sigma'}$). But

$$\omega_\sigma = \omega_{\sigma'} = K_\sigma \quad on\, \sigma \cup \sigma'$$

and then ω_σ (resp. $\omega_{\sigma'}$) achieves its minimum value in an interior point of Ω_σ (resp. $\Omega_{\sigma'}$), and is therefore constant in Ω_σ by the maximum principle. This is not possible because $\phi \not\equiv 0$.

■

3 Stability

3.1 Statement

In this section, the problem (1) is again considered. The overspecified data on the open set M of the boundary $\partial\Omega$ have been obtained by measurements, and are thus subject to errors. The stability results we are seeking at mean, roughly speaking, that *small* errors on the measurements lead to *small* perturbations on the unknown geometry. To formalize this idea, let us consider a set Σ of admissible geometries (cracks), and the mapping η defined, the *identifying* flux ϕ of the previous section being given, by :

$$\begin{array}{cccccc} \eta & : & \Sigma & \longmapsto & L^2(M) \\ & & \sigma & \longmapsto & f = u_\sigma|_M \end{array}$$

The identifiability result proved in the previous section means that this mapping is injective, and therefore, that the mapping :

$$\begin{array}{cccccc} \eta & : & \Sigma & \longmapsto & \eta(\Sigma) \\ & & \sigma & \longmapsto & f = u_\sigma|_M \end{array}$$

is invertible. The stability will be established if one proves, after having equipped Σ with an appropriate topology, that η^{-1} is continuous.

3.2 A "weak" stability result

It is easy to see that the compactness of the set of admissible geometries (with respect to the topology defined on it), together with the identifiability result, lead to the stability. This result seems to be general : it was proved for a buried crack in [12, Friedman and Vogelius], and for monotone inclusions in [1, Alessandrini]. The next theorem is devoted to this kind of result, in the situation of straight cracks emerging in a known point of the external boundary. The set Σ is equipped with the Hausdorff metric :

$$d(\sigma, \sigma') = d(T_\sigma, T_{\sigma'})$$

where T_σ (respectively $T_{\sigma'}$) is the second endpoint of the crack σ (resp. σ'), situated inside the domain Ω, and it is supposed to be compact with respect to this metric.

Theorem 2 (weak stability) *The mapping $\eta^{-1} : \eta(\Sigma) \longmapsto \Sigma$ is continuous.*

Proof : Let σ_n be a sequence of cracks in Σ such that the corresponding measured data $f_{\sigma_n} \longrightarrow f_\sigma$ in $L^2(M)$, where σ is a given crack in Σ and f_σ the corresponding measured temperature. Since Σ is a compact set, one can find some convergent subsequence $\sigma_{\alpha(n)}$ of σ_n, and let us denote by $\tilde{\sigma}$ its limit in Σ. Then, by the continuity of the solution of the direct problem (1) with respect to the data, one can easily prove the convergence $f_{\sigma_{\alpha(n)}} \longrightarrow f_{\tilde{\sigma}}$ and therefore $f_{\tilde{\sigma}} = f_\sigma$. The identifiability result (theorem 1) then gives $\sigma = \tilde{\sigma}$. σ is thus the unique adherent element to the sequence $(\sigma_n)_n$, and $\lim\limits_{n\to\infty} d(\sigma_n, \sigma) = 0$.

■

3.3 Local Lipschitz stability

3.3.1 Domain derivatives

The method presented in the following is based on the results established by Murat and Simon [14]. Consider a family of diffeomorphisms F_h, mapping $\Omega\backslash\sigma$ onto the domain $\Omega_{\sigma_h} = \Omega\backslash\sigma_h$, where σ_h is a family of cracks belonging to Σ. Following Murat and Simon, we shall chose F_h to be perturbations of the identity :

$$F_h = Id + h\,\theta$$

where $\theta = (\theta_1, \theta_2) \in W^{1,\infty}(\Omega\backslash\sigma)^2$, such that $\theta \equiv 0$ on $\partial\Omega$ and $\theta \equiv (1,0)$ in some neighbourhood of the crack tip.
For h small enough, say $h \leq h_0$, F_h is a set of diffeomorphisms, the associated set of admissible geometries being then :

$$\Sigma = \{ \quad \Omega_{\sigma_h} = (Id + h\theta)(\Omega_\sigma) \quad \} \tag{4}$$

Let then u_{σ_h} be the solution of the identification problem posed on the domain Ω_{σ_h} :

$$\left\{ \begin{array}{lcll} \Delta u_{\sigma_h} & = & 0 & in \quad \Omega_{\sigma_h} \\ \frac{\partial u_{\sigma_h}}{\partial n} & = & 0 & on \quad \sigma_h \\ \frac{\partial u_{\sigma_h}}{\partial n} & = & \phi & on \quad \partial\Omega \end{array} \right. \tag{5}$$

Notice that, as well as for equation (1), the assumption $\displaystyle\int_{\partial\Omega} \phi = 0$ is needed to insure the existence of solutions to the above problem, while the uniqueness of such a solution is obtained under the condition $\displaystyle\int_{\partial\Omega} u_{\sigma_h} = 0$.

The following result gives the first derivative - in the direction θ - of the solution of problem (1), with respect to the domain (which depends on the single parameter h since θ has been fixed).

Proposition 1 *The scalar field u^h, defined on Ω_σ by $u^h = u_{\sigma_h} \circ F_h$, can be expanded as follows :*

$$u^h = u^0 + h\,u^1 + h\,o(h) \tag{6}$$

where u^0 is the solution u_σ of problem (1), u^1 and $o(h)$ being elements of $H^1(\Omega_\sigma)$ such that $u^1|_M = 0$, and $\lim\limits_{h\to 0} o(h) = 0$. Furthermore, u^1 - which is the Lagrange derivative of u^0 with respect to the domain - is the unique solution in $H^1(\Omega)$ such that $\int_{\partial\Omega} u^1 = 0$, of the following variational problem :

$$\int_{\Omega_\sigma} \nabla u^1 \,.\, \nabla v = \int_{\Omega_\sigma} \left(\frac{\partial \theta}{\partial M}^t + \frac{\partial \theta}{\partial M} \right) \nabla u^0 \,.\, \nabla v - \int_{\Omega_\sigma} (\nabla u^0 \,.\, \nabla v)\, div\, \theta \tag{7}$$

for any $v \in H^1(\Omega_\sigma)$

Proof : It works exactly the same way than in [9], for the elasticity problem.

■

By these particular choices of F_h, one has :

$$|f - f_h|_M = |u - u_{\sigma_h}|_M = |u - u_{\sigma_h} \circ F_h|_M, \, where \quad |\cdot| \quad is\, the\, L^2 - norm$$

Thus, in order to prove a local stability result, it suffices to prove, according to equation (6), that u^1 cannot vanish on the whole M. We shall split this result into two parts : the stability with respect to the length, and the stability with respect to the angle.

3.3.2 Stability with respect to the length

Let σ be a line segment crack, with S as an endpoint on $\partial\Omega$, and let us denote by T its second endpoint (internal to the domain Ω). Let $(\sigma_h)_h$ be a family of cracks included in the line (ST), such that $|\sigma_h| = (1+h)|\sigma|$.

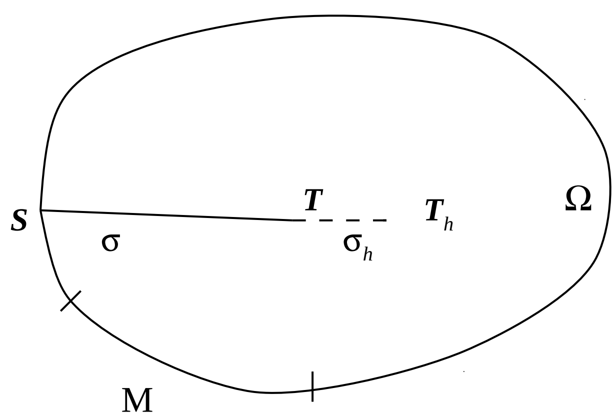

Figure 2: *Stability with respect to the length*

Theorem 3 *Let f_h be the trace of the solution u_{σ_h} of problem (5) on the boundary $\partial\Omega$. Then, under the assumption that the singular coefficient of the solution u_σ of (1) is different of zero, one has :*

$$\lim_{h\to 0} \frac{|f - f_h|_{L^2(M)}}{h} > 0$$

To prove this theorem, we need to recall a few preliminary results.
Some preliminary results : The solution u_σ of problem (1) is well known to split into two parts : a regular part u_σ^R, which belongs to $H^2(\Omega_\sigma)$, provided that the data are smooth enough, and a singular part u_σ^S [13, Grisvard] :

$$u_\sigma = u_\sigma^R + u_\sigma^S \tag{8}$$

with

$$u_\sigma^S = c\, r^{\frac{1}{2}} \sin\frac{\varphi}{2} \tag{9}$$

(r, φ) being the polar coordinates with respect to the crack tip T.

Let us now denote by $\mathcal{W}$ and $\mathcal{W}_h$ the potential energies of problems (1) and (5). We shall define the derivative of the energy, with respect to the domain (in the direction θ), by :

$$\frac{\partial \mathcal{W}}{\partial \Omega}(\Omega_\sigma)\,.\,\theta = \lim_{h\to 0}\frac{\mathcal{W}_h - \mathcal{W}}{h}$$

Then, we have the following result :

Proposition 2

$$\frac{\partial \mathcal{W}}{\partial \Omega}(\Omega_\sigma)\,.\,\theta = \int_{\partial\Omega} \phi\,.\,u^1$$

Proof : The proof is similar to the one given in [9], in the framework of elasticity, or in [10] for the Laplace equation with Dirichlet boundary conditions.

■

The next lemma relates the singularity coefficient c to the derivative of the potential energy with respect to the domain. Its proof works also the same way than in [10].

Lemma 1

$$\frac{\partial \mathcal{W}}{\partial \Omega}(\Omega_\sigma)\,.\,\theta = -\,2\,\pi\, c^2$$

Proof of the theorem : By the asymptotic expansion (6), one has :

$$\lim_{h\to 0} \frac{|f - f_h|_{L^2(M)}}{h} = |u^1|_{L^2(M)}$$

Lipschitz-stability is then proved if one can establish that $|u^1|_{L^2(M)} > 0$. Suppose this does not hold, that is $u^1 \equiv 0$ on M. By the asymtotic expansion (6), we get also $\dfrac{\partial u^1}{\partial n} \equiv 0$ on M, since the normal derivatives of u_σ and u_{σ_h} are vanishing on M. By our particular choice of θ (which is vanishing in some connected neighbourhood $\mathcal{O}$ of $\partial\Omega$), and the variational formulation (7), u^1 is harmonic in $\mathcal{O}$. It comes then by the Holmgren's unique continuation theorem that u^1 vanishes in $\mathcal{O}$, and thus that $u^1 \equiv 0\, on\, \partial\Omega$. Then, proposition 2 induces that $\dfrac{\partial \mathcal{W}}{\partial\Omega}(\Omega_\sigma)\,.\,\theta = 0$ and, as a consequence of lemma 1, that $c = 0$, which is contradictory with our hypothesis.

■

3.3.3 Stability with respect to the angle

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two open neighbourhoods of σ in Ω, such that :

$$\overline{\mathcal{V}_1} \subseteq \mathcal{V}_2 \subseteq \overline{\mathcal{V}_2} \subseteq \Omega$$

$$\overline{\mathcal{V}_1} \cap \partial\Omega = \{S\}$$

Let us now consider the family of mappings $F_h = Id + h\,\theta$, where θ is given by :

$$\theta = \left\{ \begin{array}{c} -y \\ x \end{array} \right. in\,\overline{\mathcal{V}_1} \quad ; \quad \theta = 0\,in\,\Omega_\sigma \setminus \overline{\mathcal{V}_2}$$

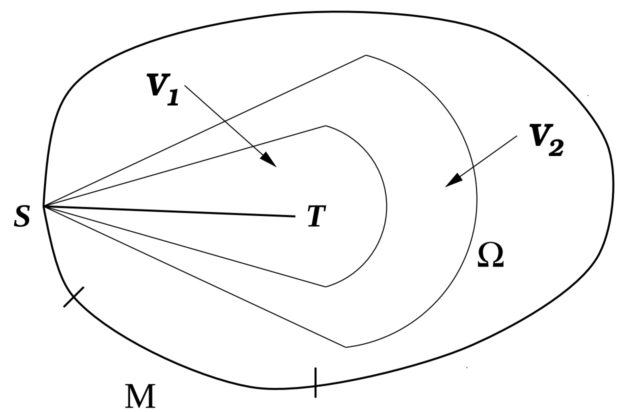

Figure 3: *Stability with respect to the angle*

where θ is smoothly extended to the whole domain Ω_σ, with regularity $W^{1,\infty}(\Omega_\sigma)$. Defining as usual $\Omega_{\sigma_h} = (Id + h\,\theta)\,\Omega_\sigma$, we have again the following result :

Theorem 4 *Let f and f_h be the traces on M of the solutions u and u_h of problems* (1) *and* (5) *posed on Ω_σ and Ω_{σ_h}. Then*

$$\lim_{h\to 0} \frac{|f - f_h|_{L^2(M)}}{h} > 0$$

Proof : The proof will split into three steps. Suppose that the result above does not hold. Then :
Step 1 : We shall prove that $u^1 \equiv 0$ on $\partial\Omega$. As for the previous subsection, proposition 1, and the asymptotic expansion (6) hold, which give :

$$\lim_{h\to 0} \frac{|f - f_h|_{L^2(M)}}{h} = |u^1|_{L^2(M)}$$

The arguments used in the proof of Theorem 3 give again here $u^1 \equiv 0$ on $\partial\Omega$.
Step 2 : Isolate the singularity

Consider a bole B_δ, with radius δ, around the crack tip T of σ, and let us denote by Ω_δ the complementary of B_δ in Ω_σ.

By proposition 1, we know that u^1 is solution of the problem (7) hereafter recalled :

$$\int_{\Omega_\sigma} \nabla u^1 \,.\, \nabla v = \int_{\Omega_\sigma} \left(\frac{\partial\theta}{\partial M}^t + \frac{\partial\theta}{\partial M} \right) \nabla u^0 \,.\, \nabla v - \int_{\Omega_\sigma} (\nabla u^0 \,.\, \nabla v)\, div\,\theta$$

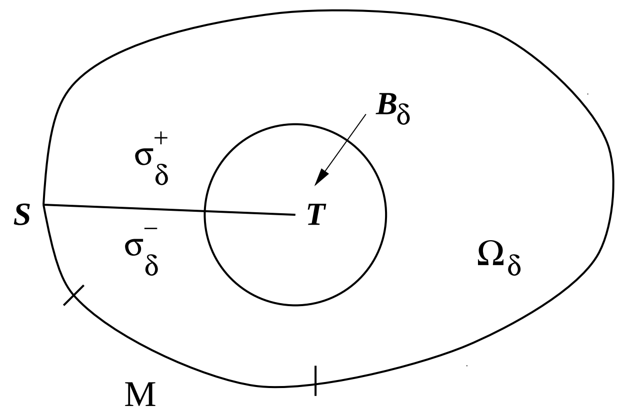

Figure 4: *Isolating the singularity*

for any $v \in H^1(\Omega_\sigma)$. Splitting the integrals on Ω_σ into two parts, the first one on Ω_δ, and the second one on B_δ, this problem is equivalent to the following :

$$\int_{\Omega_\delta} \nabla u^1 \,.\, \nabla v \;-\; \int_{\Omega_\delta} \left(\frac{\partial \theta}{\partial M}^t + \frac{\partial \theta}{\partial M} \right) \nabla u^0 \,.\, \nabla v \;+\; \int_{\Omega_\delta} (\nabla u^0 \,.\, \nabla v)\, div\, \theta +$$

$$+ \int_{B_\delta} \left\{ \nabla u^1 \,.\, \nabla v \;-\; \left(\frac{\partial \theta}{\partial M}^t + \frac{\partial \theta}{\partial M} \right) \nabla u^0 \,.\, \nabla v \;+\; (\nabla u^0 \,.\, \nabla v)\, div\, \theta \right\} \;=\; 0$$

The quantities to be integrated on B_δ belong to $L^1(\Omega_\sigma)$, and according to the fact that $\lim_{\delta \to 0} meas(B_\delta) = 0$, one also has :

$$\lim_{\delta \to 0} \left\{ \int_{B_\delta} \left[\nabla u^1 \,.\, \nabla v \;-\; \left(\frac{\partial \theta}{\partial M}^t + \frac{\partial \theta}{\partial M} \right) \nabla u^0 \,.\, \nabla v \;+\; (\nabla u^0 \,.\, \nabla v)\, div \theta \right] \right\} = 0$$

Using Green's formulae, and harmonic test functions, we derive that :

$$\lim_{\delta \to 0} \left\{ \int_{\partial \Omega} \left[u^1 \frac{\partial v}{\partial n} - (\nabla u_\sigma \,.\, \nabla v)\, \theta \,.\, n \right] + \int_{\sigma_{\delta+} \cup \sigma_{\delta-}} \left[u^1 \frac{\partial v}{\partial n} - (\nabla u_\sigma \,.\, \nabla v)\, \theta \,.\, n \right] \right.$$

$$\left. + \int_{\partial B_\delta} \left[u^1 \frac{\partial v}{\partial n} - (\nabla u_\sigma \,.\, \nabla v)\, \theta \,.\, n \right] \quad \right\} \quad = \; 0$$

$$for\, any\, v \in H^1(\Omega_\sigma)\; such\, that \quad \Delta\, v \;=\; 0 \quad in\, \Omega_\sigma$$

Since it has been proved in step 1 that $u^1 \equiv 0$ on $\partial \Omega$, the first integral vanishes. As for the second integral, it vanishes because $\theta \equiv 0$ on $\partial \Omega$. It remains :

$$\lim_{\delta \to 0} \left\{ \int_{\sigma_{\delta+} \cup \sigma_{\delta-}} \left[u^1 \frac{\partial v}{\partial n} - (\nabla u_\sigma \,.\, \nabla v)\, \theta \,.\, n \right] \right.$$

$$\left. + \int_{\partial B_\delta} \left[u^1 \frac{\partial v}{\partial n} - (\nabla u_\sigma \,.\, \nabla v)\, \theta \,.\, n \right] \quad \right\} \quad = \; 0 \tag{10}$$

$$for\, any\, v \in H^1(\Omega_\sigma) \quad such\, that \quad \Delta\, v \;=\; 0 \quad in\, \Omega_\sigma$$

Step 3 : Selecting a set of test functions

A specific sequence of test functions $(v_m)_{m\in N}$ is now selected in order to lead to a contradiction. Define the following harmonic polynomials in Ω :

$$v_m \;=\; \rho^m \,\cos(m\vartheta)$$

where (ρ, ϑ) are the polar coordinates with respect to the crack emerging point S.
Replace now v by v_m in equation (10), and let us us look at the terms integrated over B_δ. Since $u^1|_{\partial B_\delta} \in L^2(\partial B_\delta)$, as well as $\dfrac{\partial v_m}{\partial n}$, one has :

$$\left| \int_{\partial B_\delta} u^1 \, \frac{\partial v_m}{\partial n} \right| \;\le\; c_1(m)\, meas(\partial B_\delta)$$

and therefore

$$\lim_{\delta\to 0} \quad \left| \int_{\partial B_\delta} u^1 \, \frac{\partial v_m}{\partial n} \right| \;=\; 0$$

Consider at this point the splitting (8) of the solution u_σ into a singular part u_σ^S and a regular part u_σ^R, and recall that the singular part has $r^{\frac{1}{2}}$-behaviour near the crack tip (r being the distance to it), while the regular part belongs to $H^2(\Omega_\sigma)$, to derive that :

$$\left| \int_{\partial B_\delta} (\nabla u_\sigma \,.\, \nabla v_m)\,\theta\,.\,n \right| \;\le\; c_2(m)\,\delta^{\frac{1}{2}}\, meas(\partial B_\delta) \quad \forall\, m \ge 1$$

and therefore that :

$$\lim_{\delta\to 0} \quad \left| \int_{\partial B_\delta} (\nabla u_\sigma \,.\, \nabla v_m)\,\theta\,.\,n \right| \;=\; 0 \quad \forall\, m \ge 1$$

The integrals over ∂B_δ have then no contribution to the left handside of equation (10). Furthermore, $\left.\dfrac{\partial v_m}{\partial n}\right|_\sigma = \left.\dfrac{\partial v_m}{\partial \vartheta}\right|_{\vartheta=0} = \left.\dfrac{\partial v_m}{\partial \vartheta}\right|_{\vartheta=2\pi} = 0$. It comes out :

$$\lim_{\delta\to 0} \int_{\sigma_{\delta+}\cup\sigma_{\delta-}} (\nabla u_\sigma \,.\, \nabla v_m)\,\theta\,.\,n \quad = \quad \int_\sigma (\,[\nabla u_\sigma]\,.\nabla v_m\,)\,\theta\,.\,n \quad = \quad 0 \quad \forall\, v_m$$

where $[.]$ denotes the jump of the function between the brackets across the crack σ. Then :

$$\int_0^l \left[\frac{\partial u_\sigma}{\partial \rho}\right] \rho^m \quad = \quad 0 \quad \forall m \ge 1$$

which can also be written, due to the fact that the operators $[\,.\,]$ and gradient commutate [15] :

$$\int_0^l \frac{\partial}{\partial \rho}[u_\sigma]\,\rho^m \quad = \quad 0 \quad \forall m \ge 1$$

The regularity of the function $[\,u_\sigma\,]$ (actually, $u_\sigma \in H^{\frac{3}{2}-\varepsilon}(\Omega_\sigma)$, and thus its trace on the boundary σ has a $H^{1-\varepsilon}(\sigma)$-regularity), allosw us to integrate by parts and to derive :

$$\int_0^l \frac{\partial}{\partial \rho}[u_\sigma]\,\rho^m \quad = \quad -\int_0^l m\,[u_\sigma]\,\rho^{m-1} \quad + \quad [\rho^m\,[\,u_\sigma\,]]_0^l \tag{11}$$

The integrated part in the right handside of equation (11) vanishes since the crack is closed at $\rho = l$ (crack tip), and thus that $[\,u_\sigma\,]|_{\rho=l} = 0$. It follows that :

$$\int_0^l [u_\sigma]\,\rho^m \;=\; 0 \quad \forall\, m \ge 0$$

which means that $[u_\sigma]$ is L^2 - orthogonal to all the polynomials on $[0\,,\,l]$, and thus that $[u_\sigma] = 0$ on σ. This means that the crack is not detected by the given flux ϕ, which is in contradiction with the identifiability result proved in section 2.

■

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